Midterm Exam - Discrete Mathematics
B. Math. II

12 September, 2025

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 105).

(iii) You are not allowed to consult any notes or external sources for the exam, though you may use a scientific calculator.

Name: ____________________

Roll Number: ____________________

1. (15 points) Let $n \geq 5$. With justification, find the number of permutations of the set $\{1, 2, \ldots, n\}$ with exactly 5 fixed points.

Total for Question 1: 15

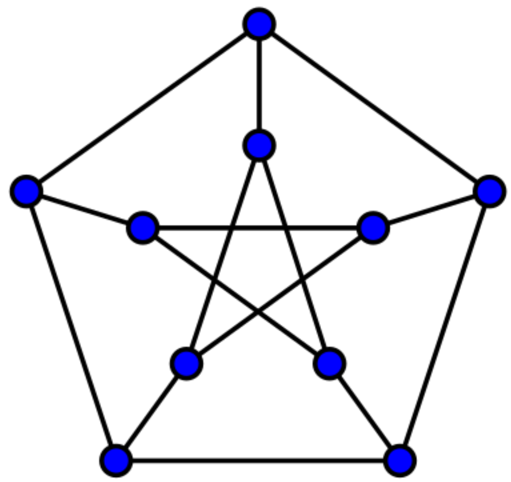
2. (15 points) Assuming that there are sufficiently many beads of colours *red* and *blue*, find the number of distinct necklaces with n beads that can be constructed.

Total for Question 2: 15

3. (15 points) Let $G = (V, E)$ be a connected finite simple graph. Prove that G has a path of length at least $\delta(G)$, where $\delta(G) := \min_{v \in V} \deg(v)$ is the minimum degree of G.

Total for Question 3: 15

4. (20 points) Prove that the Petersen graph (picture below) is not Hamiltonian.



Total for Question 4: 20

5. Let G be a simple graph on n vertices with adjacency matrix A. Let $\mathbf{e} \in \mathbb{R}^n$ be the column vector all of whose entries are equal to 1.
   (a) (10 points) Show that $\sum_{v \in V} \deg(v)^2 = \mathbf{e}^T A^2 \mathbf{e}$.
   (b) (15 points) Show that the number of (simple) 4-cycles in G is
$$\frac{1}{8}\Big(\operatorname{tr}(A^4) - \mathbf{e}^T A^2 \mathbf{e} \Big).$$

Total for Question 5: 25

6. (15 points) Show that the cycle on n vertices, C_n, has chromatic polynomial,
$$P(C_n, x) = (x-1)^n + (-1)^n (x-1).$$

Total for Question 6: 15